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Articles

Simulating normal modes and beats in a periodic inductor-capacitor circuit

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This paper approaches a periodic inductor-capacitor circuit with N-coupled loops. Kirchhoff's voltage and current laws are used in order to obtain N-coupled second-order differential equations. These equations are decoupled by using a standard discrete sine transformation. Then the voltages and currents along the circuit are written as linear combinations of harmonic waves whose normal frequencies were determined. There are N of these frequencies that depend not only on the values of the capacitance and inductance but also, on how these elements are connected. Furthermore, one presents how each normal frequency can be synchronized using external batteries that initially charge the capacitors. Additionally, not only the analytical results are presented, but also, all circuits were simulated, using the free online platform *Multisim*, to corroborate the analytical calculations. Further, the simulations present a way of exploring the complex subject of coupled oscillations without dealing with all the analytical calculations. Moreover, one explores interferences that produce beats in the oscillating voltages in the circuit. Finally, a general interpretation of how the currents flow in the circuit is presented when a single normal mode is excited.
Keywords: Periodic inductor-capacitor structures, coupled oscillations, normal modes, electrical circuit simulations.

1. Introduction

Coupled oscillations are often observed in nature, so their properties have been vastly studied. There are several applications such as the determination of the allowed frequencies of emission and absorption of electromagnetic waves by molecules in electronic spectroscopy [1]; the investigation of classical physics properties like specific heat, thermal conductivity, elasticity, etc. of crystalline solids [2, 3]; the electron-phonon interaction that leads to conventional superconductivity [4]; biological oscillators, since neural systems seem to operate on oscillatory signals [5]; computation [6]; and so on. Furthermore, it is a well-known fact that it is possible to investigate coupled oscillators by using electrical circuit systems composed of reactive elements (capacitor C and inductor L). Even a single-loop inductor-capacitor (LC) circuit presents a normal frequency (NF) because it periodically converts energy stored in a magnetic to an electric field, and vice-versa [7]. On the other hand, circuits with two or more loops will be governed by more NFs. An interesting problem is the investigation of these frequencies on periodic circuits with a large/arbitrary number of loops [9–11].

LC structures were used in early oscillator-based computers [12]. They are also commonly studied in filter theory, where they are used in radio frequency and microwave components and devices, for example [13, 14]. Furthermore, they also have been used to simulate quantum lattice models. Several works have shown that even though they have different natures, they present the same behavior because they are governed by the same dynamical equations [15–18]. This is an exciting subject of study, especially because one can explore microscopic solid-state phenomena using simple and less expensive LC circuits.

The paper is structured as follows: Section 2 reviews the simple case of two coupled electrical oscillators. The conditions for synchronous harmonic oscillations are discussed and beats are found in an inhomogeneous circuit; then in Section 3 the solution for the general case of a periodic LC circuit with N loops is obtained, moreover, the section provides a detailed description of the requirements for the synchronization of each natural vibration for $N = 3, 4$, and 5; Section 4 demonstrates the main result of the paper, namely, the general current flows in the circuit that allows the synchronization of the normal modes; and, Section 5 presents an interesting and unprecedented study of beating patterns in large homogeneous circuits. Last but not least, the free online platform *Multisim* was used to perform numerical simulations that not only validate and complement the analytical investigations but can also be used as an attractive way to explore normal modes and interference, omitting some long analytical calculations.

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2. Two Coupled Electrical Oscillation

In this section, an LC simple circuit with two loops is analyzed. First, the currents and voltages on the elements of the circuit are obtained. Since this toy system has two degrees of freedom, then it presents two normal modes of oscillation that are tuned individually. The electrical circuit considered is shown in Fig. 1.

The left (right) loop current is defined as i_1 (i_2). Furthermore, it is practical to write the currents as time derivatives of the charges q_n, i.e., $i_n = {dq_n}/{dt}$, with $n = 1, 2$. Moreover, the first and second derivatives of a function with respect to time are denoted as ${dq}/{dt} = \dot{q}$ and ${d^2q}/{dt^2} = \ddot{q}$, respectively. Throughout the entire paper, all inductors are indistinguishable with the same inductance L, in addition, the instantaneous voltage across them is $-L{di}/{dt} = -L\ddot{q}$. Mutual inductance is not considered here.

Lowercase letters are used for time-dependent variables and capital letters for fixed values. Moreover, to present a shorter notation, the time dependence on the variables is frequently omitted. The voltages on capacitors C_0, C_1, and C_2 (see Fig. 1) are $v_0 = q_1/C_0$, $v_1 = (q_2 - q_1)/C_1$, and $v_2 = -q_2/C_2$, respectively.

Kirchhoff's loop rule leads to $\frac{q_1}{C_0} = -L\ddot{q}_1 + \frac{q_2-q_1}{C_1} - L\ddot{q}_1$ and $\frac{q_2-q_1}{C_1} = -L\ddot{q}_2 - \frac{q_2}{C_2} - L\ddot{q}_2$. By taking $C_0 = C_2 = C$ and then adding and subtracting these equations one obtains $(\ddot{q}_1 \pm \ddot{q}_2) + \omega_\pm^2(q_1 \pm q_2) = 0$, where $\omega_+ = \omega_s = 1/\sqrt{2LC}$, and $\omega_- = \omega_f = \omega_s\sqrt{1 + 2C/C_1}$ are the NFs. Since $\omega_f \geqslant \omega_s$, the larger frequency is related to a smaller oscillation period and is called the fast frequency ω_f, On the other hand, ω_s is labeled as the slow one.

The above equations are simple harmonic equations whose solutions are $(q_1 + q_2) = 2Q_s \cos\omega_s t$, and $(q_1 - q_2) = 2Q_f \cos\omega_f t$, where Q_s and Q_f are two constants that are determined by the initial conditions. Solving for q_1 and q_2 leads to

$$\begin{cases} q_1(t) = Q_s \cos\omega_s t + Q_f \cos\omega_f t \\ q_2(t) = Q_s \cos\omega_s t - Q_f \cos\omega_f t. \end{cases} \tag{1}$$

It is worth pointing out that $\dot{q}_1(t = 0) = \dot{q}_2(t = 0) = 0$, i.e, the currents were chosen to be null at $t = 0$. The charges Q_s and Q_f are supplied by the external batteries that are connected to the capacitors for $t < 0$ and are disconnected from the circuit at $t = 0$, see Fig. 1. A possible battery that charges the central capacitor C_1 is omitted in this figure.

One can tune the two NFs by simply setting either $Q_s = 0$ or $Q_f = 0$. The next subsections present these conditions for synchronous harmonic oscillations in the circuit. A similar setup was discussed in Ref. [19, 20].

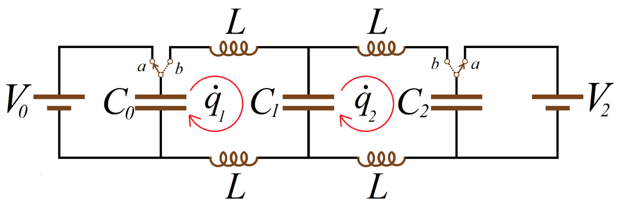

Figure 1: Two-loop circuit connected to the batteries for $t < 0$. At $t = 0$ the switch changes from "a" to "b" and the oscillations begin.

2.1. Slow frequency mode $Q_s \neq 0$ and $Q_f = 0$

To get $Q_f = 0$, the external batteries in Fig. 1 need to obey $V_2 = -V_0$, in other words, they are identical but one of them is connected with reverse polarity. Moreover, C_1 is uncharged, so $q_1 = q_2$ for all times, further

$$v_0(t) = -v_2(t) = V_0 \cos\omega_s t, \quad \text{and} \quad v_1(t) = 0 \tag{2}$$

where $V_0 = Q_s/C$.

These voltages are shown in Fig. 2-A, where the values of the used elements are $L = 1.00\,\text{mH}$ and $C = 10.0\,\mu\text{F}$, *these values were used throughout this whole paper.* For simplicity, here, one makes all capacitors identical, i.e., $C_1 = C$. Moreover, $V_0 = 5.00\,\text{V}$ in this figure. The slow NF is $\omega_s \approx 7.07 \cdot 10^3\,\text{rad/s}$ and the oscillation period is $T_s = 2\pi/\omega_s \approx 0.889\,\text{ms}$. The analytical result is compared to a simulation of the electrical circuit using the free online platform *Multisim*. The link for the simulation is [Simulation 1]. As expected, they are in complete agreement. It is worth mentioning that the analytical considerations despised any electrical resistance, therefore, it is important to set up all components in the simulation to be the closest possible to the ideal case.

Since, $v_0 = -v_2$, when the left capacitor is discharging the right one is charging, and vice-versa. Moreover, there is a pure destructive interference on C_1. In other

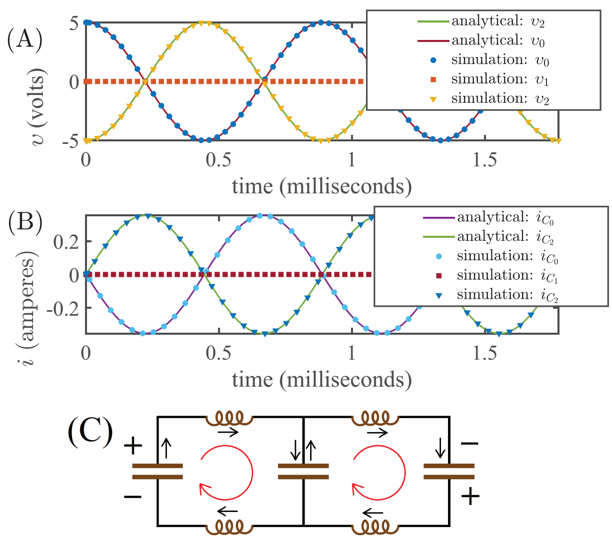

Figure 2: Slow mode in the two-loop circuit: (A) Voltage vs. time. (B) Current vs. time. (C) Current flow that is represented as $\overrightarrow{0\;1\;2}$.

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words, this system behaves as if C_1 does not exist, so there is one big loop with two identical capacitors in series and four identical inductors in series, as well. The equivalent capacitance and inductance are $C_{eq} = C/2$ and $L_{eq} = 4L$, respectively. The resonance frequency of a simple $L_{eq}C_{eq}$ circuit is $\omega_{eq} = 1/\sqrt{L_{eq}C_{eq}}$ [7], then $\omega_{eq} = \omega_s$.

As a complement, the currents on each capacitor were calculated, and they are shown in Fig. 1-B. The current on capacitors C_n is given by $i_{C_n} = C_n \dot{v}_n$ ($n = 1, 2$, and 3), so $i_{C_0} = -i_{C_2} = -\omega_s V_0 \sin \omega_s t$, and $i_{C_1} = 0$. At $t = 0$, all currents are null, and then during the first fourth of the oscillation period ($0 < t < T_s/4$), the positively charged capacitor (C_0 on the left) starts to discharge (positive current). This current goes straight to the negatively charged capacitor C_2, then the current is negative because C_2 is receiving current downwards on its superior plate. One shows the current flows on Fig. 2-C where both loops have the same clockwise current, then the current on the middle capacitor is always null. Furthermore, one represents this situation as $\omega_1 : \overrightarrow{0\ 1\ 2}$, since the current from capacitor C_0 goes straight to capacitor C_2 without flowing any current to capacitor C_1.

2.2. Fast frequency mode $Q_f \neq 0$ and $Q_s = 0$

To obtain $Q_s = 0$, the batteries on C_0 and C_2 are connected with the same polarity, i.e., $V_2 = V_0$. Moreover, $C_1 = C$ is used in this subsection. Then $q_1 = -q_2 = Q_f \cos \omega_f t$ and the voltages on the capacitors are

$$v_0(t) = v_2(t) = V_0 \cos \omega_f t = -v_1(t)/2, \tag{3}$$

where $Q_f = CV_0$.

The above equations, for $V_0 = 5.00\,\mathrm{V}$ together with the simulated data, see [Simulation 2], are presented in Fig. 3-A. The fast NF is $\omega_f \approx 12.3 \cdot 10^3\,\mathrm{rad/s}$ and the oscillation period is $T_f \approx 0.511\,\mathrm{ms}$. In addition, Fig. 3-B shows the current intensities on each capacitor. Since $v_0 = v_2$ and $i_{C_0} = i_{C_2} = -\omega_s V_0 \sin \omega_s t$, the loop currents have opposite directions, see Fig. 3-C, then the external capacitors are always discharging/charging together and complete constructive interference charges/discharges C_1 because $v_1 = -2v_{0(2)}$ and $i_{C_1} = -2i_{C_{0(2)}}$. This normal mode can be understood as if C_1 is being equally used by both loops so, this capacitor can be realized as if it was split in the middle and half of its capacitance is used by each loop. Then, there are two capacitor (C and $C/2$) in series so, $C_{eq} = C/3$ and the total inductance is $L_{eq} = 2L$ then, $\omega_{eq} = \omega_f$. Since the currents flowing from both C_0 and C_2 go to capacitor C_2 one represents this situaion as $\omega_2 : \overline{0\ \overset{\times}{1}\ 2}$.

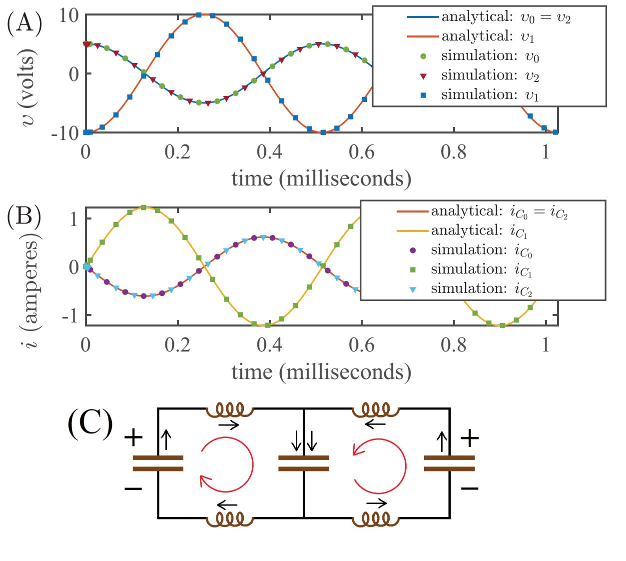

Figure 3: Fast mode in the two-loop circuit: (A) Voltage vs. time. (B) Current vs. time. (C) The current flow that can be illustrated as $\overline{0\ \overset{\times}{1}\ 2}$.

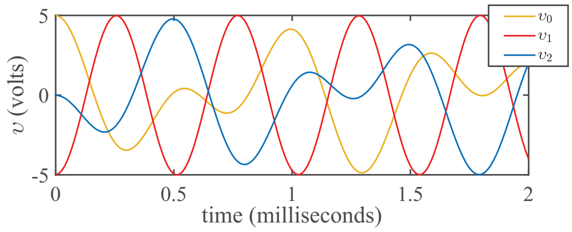

Figure 4: Two-loop circuit: timbre pattern.

2.3. Timbre

In this subsection, it was investigated signals that are a combination of both fast and slow harmonics. The resulting oscillatory voltage is no longer a simple cosine but a linear combination of the two NFs (Equation 1). By imposing that $Q_s = Q_f = CV_0/2$ the voltages-time evolution becomes

$$\begin{cases} v_0(t) = \frac{V_0}{2}\left(\cos \omega_s t + \cos \omega_f t\right) \\ v_1(t) = -\frac{C}{C_1} V_0 \cos \omega_f t \\ v_2(t) = -\frac{V_0}{2}\left(\cos \omega_s t - \cos \omega_f t\right) \end{cases}. \tag{4}$$

The voltage on each capacitor for this situation is presented in Fig. (4), where $C = C_1$, $V_0 = -V_1 = 5.00\,\mathrm{V}$, and $V_2 = 0$ were used. This case was also simulated, see [Simulation 3], but one refrained from showing any simulated data to keep the figure less busy. The simulated data were in agreement with the analytical results. As expected from Equation (4), capacitor C_1 oscillates with the fast harmonic frequency. On the other hand, C_0 and C_2 present more complex behaviors, the two simple harmonic voltages (with the same amplitude) interfere to create a timbre pattern.

2.4. Beats

The final part of this section investigates an interference pattern called beating. The system starts with the left capacitor charged, and the one on the right is uncharged. As time goes the charge is exchanged back and forth between these two capacitors. To accomplish this situation, the normal mode oscillations $Q_s \cos\omega_s t$ and $Q_f \cos\omega_f t$ have to have the same amplitudes, therefore, Equation (4) is still valid. Moreover, the two NFs need to be close to each other, since, $\omega_f = \omega_s\sqrt{1+2C/C_1}$, the condition $C \ll C_1$ leads to $\omega_s \approx \omega_f$. In this situation, it is useful to define the average of the two NFs and the half of the difference in these frequencies as

$$\bar{\omega} = \frac{\omega_s + \omega_f}{2} \quad \text{and} \quad \delta = \frac{\omega_f - \omega_s}{2}. \tag{5}$$

Therefore, the larger and the smaller frequencies are $\bar{\omega} \approx \omega_f \approx \omega_s$ and $\delta \approx \omega_s C/2C_1$, respectively.

By using standard trigonometric relations one obtains the following results:

$$\begin{cases} \upsilon_0(t) = V_0\cos(\delta t)\cos(\bar{\omega}t) \\ \upsilon_2(t) = -V_0\sin(\delta t)\sin(\bar{\omega}t) \end{cases}. \tag{6}$$

So, the above expressions represent fast oscillations with frequency $\bar{\omega}$ [$\cos(\bar{\omega}t)$ for υ_0 and $\sin(\bar{\omega}t)$ for υ_2] modulated by a slow oscillation with frequency δ. Moreover, the enveloping curves for the left and right capacitors are $V_0\cos(\delta t)$ for υ_0 and $-V_0\sin(\delta t)$ for υ_2.

Results for $C_1 = 10\,C = 100\,\mu$F, $V_0 = 5.00\,$V, $V_1 = -\frac{C}{C_1}V_0 = 0.500\,$V, and $V_2 = 0.00\,$V are shown in Fig. (5), for the usual values of L and C. The energy is initially stored in C_0, nonetheless, the two-loops are weakly coupled, and then with time, the energy is traded back and forth between C_2 and C_0. So, the smaller NF is again $\omega_s \approx 7.07 \times 10^3$ rad/s but the larger one changes to $\omega_f \approx 7.76 \times 10^3$ rad/s. The average of these two frequencies is $\bar{\omega} \approx 7{,}41 \times 10^3$ rad/s, and it gives the fast oscillations observed in the figure. The voltages are represented by the circle symbols. These results were also simulated using the *Multisim* platform, see [Simulation 4]. The

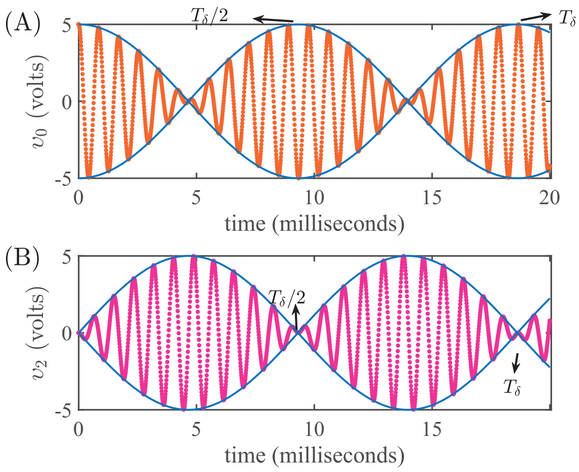

Figure 5: Two-loop circuit with beating patterns on the external capacitors.

results for υ_1 are not presented because it is only a small simple harmonic oscillation. Last but not least, the lines in Fig. (5) are the modulations (envelopes). They present a slow frequency of $\delta = 0.337 \times 10^3$ rad/s whose equivalent time period is $T_\delta = 2\pi/\delta \approx 0.0182\,$s.

The mathematics of this problem is discussed in Ref. [8] for the case of mechanic-coupled oscillations.

3. General N-loop Electrical Circuit

The electrical circuit discussed in this section is shown in Fig. 6. The periodic structure has a ladder geometry with an arbitrary N number of loops. The capacitors are all identical (same capacitance C). Let i_j (with $j = 1, \ldots, N$) be the loop current in the j-th loop, and q_j is the charge associated with this current, i.e., $i_j = \dot{q}_j$. By applying Kirchhoff's laws for this loop, one has that

$$2LC\ddot{q}_j + 2q_j - q_{j-1} - q_{j+1} = 0, \tag{7}$$

where the boundary condition $q_0 = 0 = q_{N+1}$ needs to be satisfied. Moreover, the voltage across the j-th capacitor ($j = 0, 1, \ldots, N$) is given by

$$\upsilon_j = (q_{j+1} - q_j)/C. \tag{8}$$

Equation 7 represents a set of N-coupled homogeneous second-order differential equations with constant coefficients. The charges $\{q_j\}$ are decoupled by *diagonalizing* this set of equations with the following canonical discrete sine transformation

$$q_j = \sum_{n=1}^{N} p_n \sin(\theta_n j) \tag{9}$$

where $\theta_n = n\pi/(N+1)$. The inverse transformation is $p_n = 2(N+1)^{-1}\Sigma_j q_j \sin(\theta_n j)$, and it is a consequence of the orthogonality and completeness of the set of functions $\{\sin\theta_n j\}$, i.e., $\Sigma_j \sin(\theta_n j)\sin(\theta_m j) = (N+1)\delta_{n,m}/2$, where $\delta_{n,m}$ is a Kronecker delta. Notice that, the above definition satisfies que requirement $q_0 = q_{N+1} = 0$.

The substitution of Equation (9) into Equation (7) leads to $\ddot{p}_n(t) + \omega_n^2 p_n(t) = 0$, where

$$\omega_n = 2\,\omega_0 \sin(\theta_n/2), \tag{10}$$

are the NFs and $\omega_0 = 1/\sqrt{2LC}$. Since p_n obeys a simple harmonic equation then, its solution is $p_n(t) = P_n \cos\omega_n t$, where P_n is a constant. Moreover $\dot{p}_n(t=0) = 0$ means $\dot{q}_j(t=0) = 0$, i.e., all currents are zero at $t = 0$.

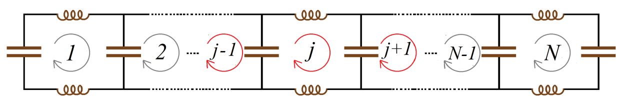

Figure 6: N-loops circuit. The capacitance C and inductance L have the same values for all elements.

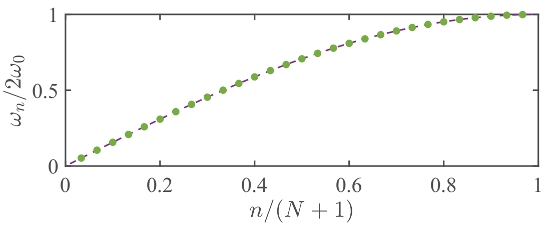

Figure 7: The NF ω_n vs. n. The circles represent a system with $N = 29$ and the dashed line is the limit $N \to \infty$.

Finally, the charge q_j can be written as a superposition of the p_n oscillations as

$$q_j(t) = \sum_{n=1}^{N} P_n \sin(\theta_n j) \cos \omega_n t. \tag{11}$$

Last but not least, the behavior of the NFs as a function of n is presented in Fig. 7. The fast and slow frequencies are $\omega_f = \omega_N$ and $\omega_s = \omega_1$, respectively. In the larger N limit $\omega_f \to 2\,\omega_0$ and $\omega_s \to 0$.

The remaining of this section explores in detail the normal modes for the particular cases $N = 3, 4$, and 5.

3.1. $N = 3$: Three coupled electrical oscillations

For $N = 3$, the loop charges given by Equation (11) can be written on a matrix structure as

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} P_1 \cos \omega_1 t \\ P_2 \cos \omega_2 t \\ P_3 \cos \omega_3 t \end{bmatrix}, \tag{12}$$

and the NFs are $\omega_1 = \omega_0\sqrt{2-\sqrt{2}}$, $\omega_2 = \omega_0\sqrt{2}$, and $\omega_3 = \omega_0\sqrt{2+\sqrt{2}}$ [Equation (10)]. For the established values of L and C, the NFs become $\omega_1 \approx 5.41 \cdot 10^3\,\text{rad/s}$, $\omega_2 = 10.0 \cdot 10^3\,\text{rad/s}$, and $\omega_3 \approx 13.1 \cdot 10^3\,\text{rad/s}$, additionally, the oscillation periods are $T_1 \approx 1.16\,\text{ms}$, $T_2 \approx 0.628\,\text{ms}$, and $T_3 \approx 0.481\,\text{ms}$. Moreover, these normal oscillations are tuned by setting two constants, P_1, P_2, or P_3, to zero and keeping only one non-null. Each case is discussed in detail below.

(a) Slow mode $\omega_s = \omega_1$: For example, for $P_2 = P_3 = 0$ the voltage on the capacitors [see Equation (8)] are $v_0 = (P_1/\sqrt{2}C)\cos\omega_1 t$, $v_1 = [P_1(1 - 1/\sqrt{2})/C]\cos\omega_1 t$, $v_2 = -v_1$, and $v_3 = -v_0$. Setting $P_1 = \sqrt{2}C\bar{V}$ leads to

$$\begin{cases} v_0 = \bar{V}\cos\omega_1 t = -v_3 \\ v_1 \approx 0.414\,\bar{V}\ \cos\omega_1 t \approx -v_2. \end{cases} \tag{13}$$

The top panel in Fig. 8 presents the behavior of these voltages, where $\bar{V} = 1.00\,\text{V}$ (this value is used in the rest of the paper). For $0 < t < T_1/4$, both capacitors on the left have their top plates positively charged (circles and squares), and the ones on the right have charges of the same magnitudes and opposite signs (upward and downward triangles). Then, the charges flow from the left to the right of the circuit. This situation is depicted in the top panel in Fig. 9, the values on top of the capacitor are their initial voltage, in volts. This mode can be interpreted as the two flows of currents represented by the large and small arrows. In a way, the external capacitors exchange charge only between them, and analogously, the two capacitors in the middle only transfer charge to each other. Since there is one current flow from capacitor C_0 to C_3 and another one from capacitor C_1 to C_2, this mode is denoted as ω_1 : $\overline{0\,\overrightarrow{1\,2}\,3}$. For the simulated circuit, [Simulation 5], to reduce the number of components, the pair of inductors $L = 1.00\,\text{mH}$ on each unit cell was replaced by a single inductor $L' = 2.00\,\text{mH}$.

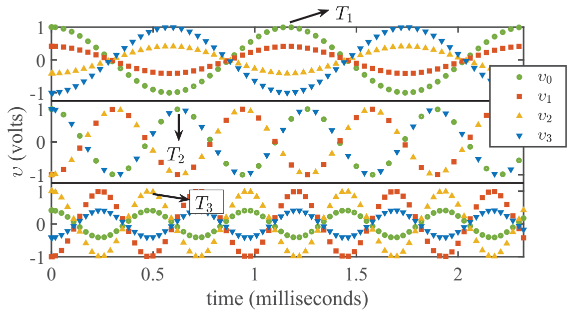

Figure 8: Normal modes oscillations for $N = 3$. From top to bottom: voltage on the capacitor for the slow, intermediate, and fast modes.

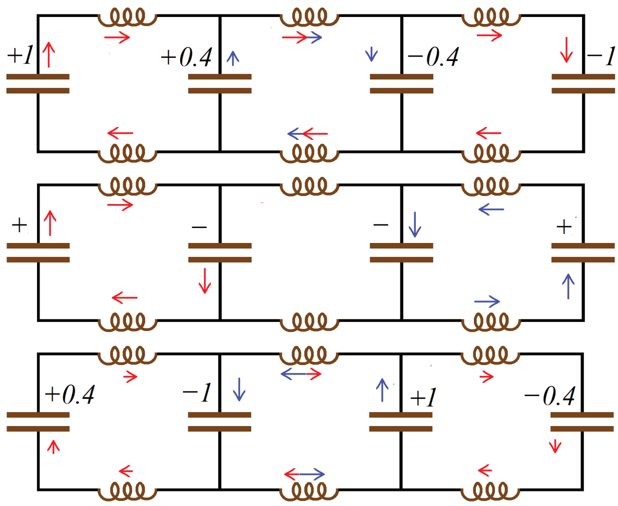

Figure 9: Current flows for $N = 3$ loops. From top to bottom, the circuit is synchronized with the frequencies ω_1, ω_2, and ω_3, respectively.

(b) Intermediate mode ω_2: The system starts with $P_1 = P_3 = 0$ and $P_2 = C\bar{V}$, so only ω_2 is presented. In this case, the voltage across the capacitors are

$$v_0 = v_3 = -v_1 = -v_2 = \bar{V}\ \cos\omega_2 t. \tag{14}$$

These four voltages are shown in the middle panel in Fig. 8. As expected, this mode has a shorter oscillation period than the previous case. The current flows (for $0 < t < T_2/4$) are also presented in the middle panel in Fig. 9. This is a much simpler situation; the currents only flow in the two external loops, in fact, the currents across the middle inductors are always zero. Since all elements are in series in the external loops, then $C_{eq} = C/2$ and $L_{eq} = 2L$. So, the resonance frequency $\omega_{eq} = 1/\sqrt{LC}$ is equal to ω_2. The notation $\omega_2 : \overrightarrow{0\,1}\,\vdots\,\overleftarrow{2\,3}$ represents the current flows in this situation because the current from C_0 goes to C_1 at the same time that the current from C_3 goes directly to C_2. The broken vertical bar stresses that there is no current flowing from the left capacitors (C_0 and C_1) to the capacitors on the right side (C_2 and C_3).

See the simulation that agrees with the analytical results on [Simulation 6].

(c) Fast mode $\omega_f = \omega_3$: This situation was accomplished by setting $P_1 = P_2 = 0$ and $P_3 = (1 + 1/\sqrt{2})C\bar{V}$, so the voltages are:

$$\begin{cases} v_0 \approx 0.414\,\bar{V}\,\cos\omega_3 t \approx -v_3 \\ v_1 = -\bar{V}\,\cos\omega_3 t = -v_2. \end{cases} \tag{15}$$

The results are in the bottom panels of Figs. 8 and 9. The interpretation for this case is similar to the slow mode. There are two current flows, one between the external capacitors and the other between the internal ones. Different from the slow (ω_1) situation, the current flows have opposite directions here. This situation is represented as $\omega_3 : \overrightarrow{0\,\overleftarrow{1\,2}\,3}$ because the charge from C_0 goes straight to C_3 at the same time that charge flows from C_2 to C_1. See [Simulation 7] for the simulation.

(d) Reobtaining the frequencies ω_1 and ω_3: This subsection concludes by using a different approach to obtain the synchronous solutions for the slow (ω_1) and fast (ω_3) modes. In the top panel in Fig. 9, the current in the central unit cell is labeled as i_c and the current in the big external loop is i_e. Therefore, they obey the following equation

$$\begin{bmatrix} q_e \\ q_c \end{bmatrix} + LC \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \ddot{q}_e \\ \ddot{q}_c \end{bmatrix} = 0. \tag{16}$$

Finally, the eigenvalues of the above 2×2 matrix are $LC(2 \pm \sqrt{2})$ and they lead to the frequencies ω_1 and ω_3, as expected.

3.2. $N = 4$: Four coupled electrical oscillations

For $N = 4$ the NF modes can be synchronized as follow:

- $\omega_1 = [(3 - \sqrt{5})/2]^{1/2}\omega_0 \to \sqrt{5 - \sqrt{5}}P_1 = \sqrt{8}C\bar{V}$ and $P_2 = P_3 = P_4 = 0$, then

$$\begin{cases} v_0 = -v_4 = \bar{V}\cos\omega_1 t \\ v_1 = -v_3 = \frac{(\sqrt{5}-1)}{2}\,\bar{V}\cos\omega_1 t \approx 0.618\,\bar{V}\cos\omega_1 t \\ v_2 = 0; \end{cases} \tag{17}$$

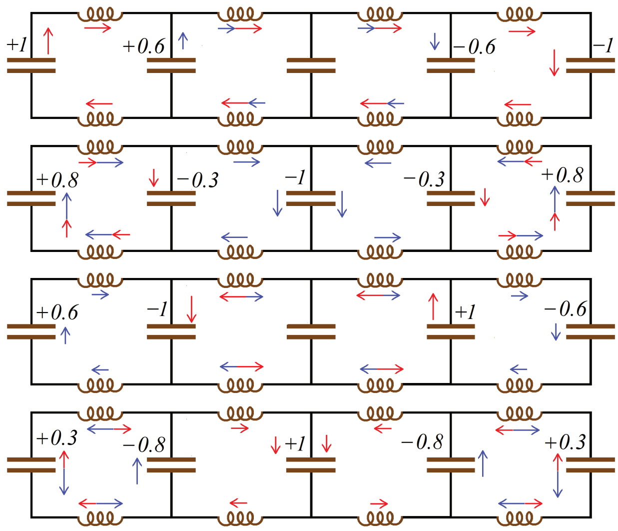

Figure 10: Currents flow in a system with $N = 4$ loops. From top to bottom, the excited NFs are ω_1, ω_2, ω_3, and ω_4.

- $\omega_2 = [(5 - \sqrt{5})/2]^{1/2}\omega_0 \to \sqrt{5 - \sqrt{5}}P_2 = \sqrt{2}C\bar{V}$ and $P_1 = P_3 = P_4 = 0$, so

$$\begin{cases} v_0 = v_4 = \frac{(1+\sqrt{5})}{4}\,\bar{V}\cos\omega_2 t \approx 0.809\,\bar{V}\cos\omega_2 t \\ v_1 = v_3 = -\frac{(\sqrt{5}-1)}{4}\,\bar{V}\cos\omega_2 t \approx -0.309\,\bar{V}\cos\omega_2 t \\ v_2 = -\bar{V}\cos\omega_2 t; \end{cases} \tag{18}$$

- $\omega_3 = [(3 + \sqrt{5})/2]^{1/2}\omega_0 \to \sqrt{5 + 2\sqrt{5}}P_3 = 2C\bar{V}$ and $P_1 = P_2 = P_4 = 0$, then

$$\begin{cases} v_0 = -v_4 = \frac{(\sqrt{5}-1)}{2}\,\bar{V}\cos\omega_3 t \approx 0.618\,\bar{V}\cos\omega_3 t \\ v_1 = -v_3 = -\bar{V}\cos\omega_3 t; \\ v_2 = 0; \end{cases} \tag{19}$$

- $\omega_4 = [(5 + \sqrt{5})/2]^{1/2}\omega_0 \to \sqrt{5 + \sqrt{5}}P_4 = \sqrt{2}C\bar{V}$ and $P_1 = P_2 = P_3 = 0$, so

$$\begin{cases} v_0 = v_4 = \frac{(\sqrt{5}-1)}{4}\,\bar{V}\cos\omega_4 t \approx 0.309\,\bar{V}\cos\omega_4 t \\ v_1 = v_3 = \frac{-(\sqrt{5}+1)}{4}\,\bar{V}\cos\omega_4 t \approx -0.809\,\bar{V}\cos\omega_4 t \\ v_2 = \bar{V}\cos\omega_4 t. \end{cases} \tag{20}$$

The current flows for each situation are presented in Fig. 10. One summarizes these flows as $\omega_1 : \overrightarrow{0\,\overrightarrow{1\,2\,3}\,4}$, $\omega_2 : \overrightarrow{\overrightarrow{0\,1}\,\,}\!\overset{\times}{2}\!\overleftarrow{\,\,\overleftarrow{3\,4}}$, $\omega_3 : \overrightarrow{0\,\overleftarrow{1\,2\,3}\,4}$, and $\omega_4 : \overleftarrow{\overleftarrow{0\,1}\,\,}\!\overset{\times}{2}\!\overrightarrow{\,\,\overrightarrow{3\,4}}$. The simulations are avaliable on: [Simulation 8]; [Simulation 9]; [Simulation 10]; [Simulation 11]. Moreover, it is possible to reobtain the above solutions by following similar steps to the ones used to obtain Equation (16) and then solving 2×2 matrices.

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3.3. $N = 5$: Five coupled electrical oscillations

The NFs, given by Equation (10), are

ω_1/ω_0	ω_2/ω_0	ω_3/ω_0	ω_4/ω_0	ω_5/ω_0
$\sqrt{2-\sqrt{3}}$	1	$\sqrt{2}$	$\sqrt{3}$	$\sqrt{2+\sqrt{3}}$

The slowest mode $\omega_s = \omega_1$ is synchronized by setting the normalization constants to $P_1 = 2C\bar{V}$ and $P_2 = P_3 = P_4 = P_5 = 0$. The next frequency ($\omega_2$) is adjusted with $P_2 = -\frac{2}{\sqrt{3}}C\bar{V}$ and $P_1 = P_3 = P_4 = P_5 = 0$. Furthermore, if the set of constants are $P_3 = C\bar{V}$ and $P_1 = P_2 = P_4 = P_5 = 0$, then the frequency ω_3 is synchronized. The frequency ω_4 is obtained when the constants are $P_4 = \frac{1}{\sqrt{3}}C\bar{V}$ and $P_1 = P_2 = P_3 = P_5 = 0$. Last but not least, setting $P_1 = P_2 = P_3 = P_4 = 0$ and $P_5 = (4-2\sqrt{3})C\bar{V}$ leads to the emergence of the fastest mode $\omega_f = \omega_5$. Therefore, each one of the NFs can be individually excited by setting up the following initial voltages.

	ω_1	ω_2	ω_3	ω_4	ω_5
$v_0(0)$	1	-1	1	$1/2$	$2-\sqrt{3}$
$v_1(0)$	$\sqrt{3}-1 \approx 0.732$	0	-1	-1	$-(\sqrt{3}-1)$
$v_2(0)$	$2-\sqrt{3} \approx 0.268$	1	-1	$1/2$	1
$v_3(0)$	$-(2-\sqrt{3})$	1	1	$1/2$	-1
$v_4(0)$	$-(\sqrt{3}-1)$	0	1	-1	$\sqrt{3}-1$
$v_5(0)$	-1	-1	-1	$1/2$	$-(2-\sqrt{3})$

In the above table, the voltage value $\bar{V}$ was again taken as 1.00 V.

Fig. 11 displays the time evolution of the voltages and Fig. 12 shows the current flows. The periods of oscillations are $T_1 \approx 1.72$ ms, $T_2 \approx 0.889$ ms, $T_3 \approx 0.628$ ms, $T_4 \approx 0.513$ ms,and $T_5 \approx 0.460$ ms. The most simple cases are $n = 2, 3$ and 4. For $n = 2$ and $n = 4$, there is no flow of current between the left and right parts of the circuit. Thus, the system behaves as two independent two-loop circuits. Further, these two modes are identical to the ones discussed in Subsections 2.1 and 2.2 for $N = 2$. On the other hand, the mode $n = 3$ behaves like three independent one-loop circuits, as observed in the $N = 3$ case. The currents flow can be viewed (for $0 < t < T_n/4$) as $\omega_1 : \overrightarrow{0\;\overrightarrow{1\;\overrightarrow{2\;3}\;4}\;5}$, $\omega_2 : \overleftarrow{0\;1\;2}\,¦\,\overrightarrow{3\;4\;5}$, $\omega_3 : \overrightarrow{0\;1}\,¦\,\overleftarrow{2\;3}\,¦\,\overrightarrow{4\;5}$, $\omega_4 : \overrightarrow{0}\;1\;\overleftarrow{2}\,¦\,\overrightarrow{3}\;4\;\overleftarrow{5}$, and $\omega_5 : \overrightarrow{0\;\overleftarrow{1\;\overrightarrow{2\;3}\;4}\;5}$. Moreover, the frequency ω_3 that presents three decoupled loops can be understood in a general way as $\omega_3 : \overrightarrow{0\;\overleftarrow{1\;\overleftarrow{2\;3}\;4}\;5}$. The numerical simulation, for the case ω_3, is available on [Simulation 12].

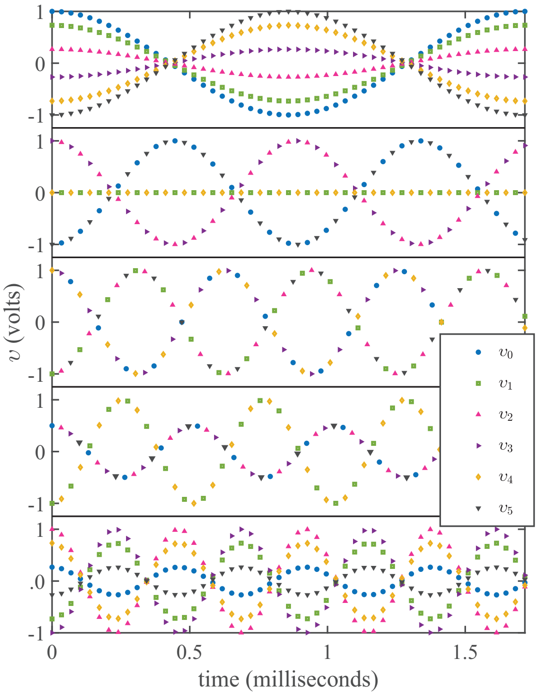

Figure 11: Voltages for an $N = 5$ circuit. From top to bottom, one shows the results from the slowest to the fastest mode.

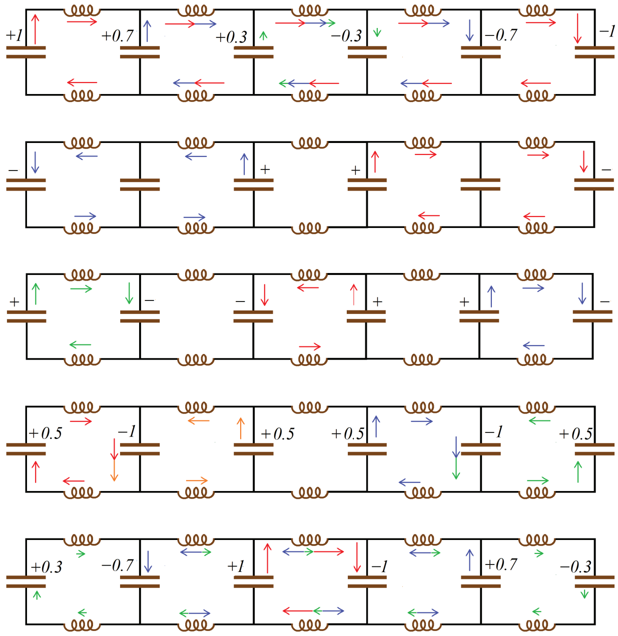

Figure 12: Currents flow in a system with $N = 5$ loops. From top to bottom the natural frequency ω_1, ω_2, ω_3, ω_4, and ω_5 was excited.

4. General Current Flow

In general, to synchronize all capacitors oscillating with the same frequency ω_m, the initial conditions are $P_m \neq 0$ and $P_n = 0$ for all $n \neq m$. So, by using Equation (8) and (11), one has that

$$v_j = P_m C^{-1}\left[\sin\theta_m(j+1) - \sin\theta_m j\right]\cos\omega_m t, \qquad (21)$$

further, it is easy to see that

$$\sum_{j=0}^{N} v_j = 0$$

and

$$v_{N-j} = (-)^m v_j.$$

These properties lead to significant results.

(a) odd m case: In this situation, $v_{N-j} = -v_j$, and then the current from the j-th capacitor will flow to the $(N-j)$-th capacitor. The current flow behavior for odd and even N is quite similar. See the graphics for the NFs ω_1, ω_3 or ω_5 on Figs. 9 and 11 for examples of $N = 3$ and $N = 5$. Bellow, as another example, it is shown the current flows for $N = 7$

$$\omega_1: \quad 0\ 1\ 2\ 3\ 4\ 5\ 6\ 7$$

$$\omega_3: \quad 0\ 1\ 2\ 3\ 4\ 5\ 6\ 7$$

$$\omega_5: \quad 0\ 1\ 2\ 3\ 4\ 5\ 6\ 7$$

$$\omega_7: \quad 0\ 1\ 2\ 3\ 4\ 5\ 6\ 7.$$

On the other hand, if N is even, the cases discussed in the past Sections are in Figs. 2 and 10, where it is worth pointing out that $v_{N/2} = 0$ for all times. E.g, for $N = 6$, when the odd frequencies (ω_1, ω_3 and ω_5) are individually excited, the current flows are

$$\omega_1: \quad 0\ 1\ 2\ 3\ 4\ 5\ 6,$$

$$\omega_3: \quad 0\ 1\ 2\ 3\ 4\ 5\ 6,$$

and

$$\omega_5: \quad 0\ 1\ 2\ 3\ 4\ 5\ 6.$$

(b) even m case: if m and N are even, then $v_{N-j} = v_j$, and

$$\sum_{j=0}^{N/2-1} v_j + \frac{v_{N/2}}{2} = 0.$$

In terms of the current flow, the capacitor $j = N/2$ is symmetrically shared with the left and right sides of the circuit. E.g., for $N = 6$, the m-even modes behave as

$$\omega_2: \quad 0\ 1\ 2\ \ 3\ 4\ 5\ 6,$$

$$\omega_4: \quad 0\ 1\ 2\ \ 3\ 4\ 5\ 6,$$

and

$$\omega_6: \quad 0\ 1\ 2\ \ 3\ 4\ 5\ 6.$$

Furthermore, if m is even and N is odd then, the $(N+1)/2$ loop divides the system into two halves. On the other hand, $\dot{q}_{\frac{N+1}{2}} \propto \sin\left(\frac{\pi m}{2}\right) = 0$ so, the system behaves like two decoupled subcircuits with $\frac{N-1}{2}$ loops each. If $\frac{N-1}{2}$ is even, the two subcircuits fit the above description, otherwise, they each decouple into two smaller circuits with $\frac{N-3}{4}$ loops. The divisions keep going until a circuit with an even number of unit cells is obtained. A further example, for $N = 7$, is provided below

$$\omega_2: \quad 0\ 1\ 2\ 3 \,\vdots\, 4\ 5\ 6\ 7$$

$$\omega_4: \quad 0\ 1 \,\vdots\, 2\ 3 \,\vdots\, 4\ 5 \,\vdots\, 6\ 7$$

$$\omega_6: \quad 0\ 1\ 2\ 3 \,\vdots\, 4\ 5\ 6\ 7,$$

notice that the circuit splits into two halves with three loops each then the current flows are equal to the ones observed in Fig. 9.

Simplified flows: Last but not least, even though the above description is general, sometimes there are simpler interpretations, as one has already discussed in the final part of Section 3.3 for $N = 5$ and ω_3. Something similar happens to $N = 8$, where the general current flows for the frequencies ω_3 and ω_6 are

$$\omega_3: \quad 0\ 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8$$

and

$$\omega_6: \quad 0\ 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8.$$

On the other hand, they can be simplified as

$$\omega_3: \quad 0\ 1\ 2 \,\vdots\, 3\ 4\ 5 \,\vdots\, 6\ 7\ 8$$

$$\omega_6: \quad 0\ 1\ 2 \,\vdots\, 3\ 4\ 5 \,\vdots\, 6\ 7\ 8,$$

respectively. So, there are three decoupled loops with two unit cells each, just like in the slow and fast frequencies for $N = 2$.

Furthermore, the decoupled single-loops that happened for $N = 5$ and ω_3, see Fig. 11, and for $N = 3$ with $m = 1$ [Fig. 9] will happen when both N and $m = \frac{N+1}{2}$ are odd. In this case, $v_j \propto \cos(\pi j/2) - \sin(\pi j/2)$, therefore, $v_j \propto (-)^{\frac{j}{2}}$ when j is even and $v_j \propto (-)^{\frac{j-1}{2}}$ if j is odd. The graphic interpretation would be

$$\overrightarrow{+-} \,\vdots\, \overleftarrow{-+} \,\vdots\, \overrightarrow{+-} \,\vdots\, \overleftarrow{-+} \,\vdots\, \overrightarrow{+-} \cdots$$

moreover, the value of this NF is always $\omega_{\frac{N+1}{2}} = \sqrt{2}\omega_0$.

5. Beating in a $N = 4$ System

In Section 2.4 it was observed that it is possible to tune a beating pattern for $N = 2$ in an inhomogeneous circuit (capacitors with different values). The

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interference between two oscillating signals of the same amplitude will become a beating when the signals have similar frequencies. Moreover, one can notice that as N increases, the difference between two consecutive NFs decreases, see Equation (10) and Fig. 7. Additionally, in general, the two closest NFs are the two largest ones. Therefore, one can expect beatings to happen in sufficiently large circuits.

Even though $N = 4$ can be viewed as a small number the difference between the frequencies ω_3 and ω_4 is small enough to produce the alternating constructive and destructive interference between the voltages that characterizes a beat. The initial condition $P_1 = P_2 = 0$ makes Equation (11), for $j = 1$, become $q_1 = P_3 \sin\theta_3 \cos\omega_3 t + P_4 \sin\theta_4 \cos\omega_4 t$. The voltage across C_0 is $\upsilon_0 = q_1/C$, then setting $P_3 \sin\theta_3 = C\bar{V} = P_4 \sin\theta_4$ imposes that the oscillations $\cos\omega_3 t$ and $\cos\omega_4 t$ have the same voltage $\bar{V}$. In this case, the voltages become

$$\begin{cases} \upsilon_0 = \bar{V}(\cos\omega_3 t + \cos\omega_4 t) \\ \upsilon_1 = -\frac{\bar{V}}{2}[(1+\sqrt{5})\cos\omega_3 t + (3+\sqrt{5})\cos\omega_4 t] \\ \upsilon_2 = \bar{V}(1+\sqrt{5})\cos\omega_4 t \\ \upsilon_3 = \frac{\bar{V}}{2}[(1+\sqrt{5})\cos\omega_3 t - (3+\sqrt{5})\cos\omega_4 t] \\ \upsilon_4 = \bar{V}(-\cos\omega_3 t + \cos\omega_4 t) \end{cases} . \quad (22)$$

The behaviors of the above equations are depicted in Fig. 13 for $\bar{V} = 1.00\,\mathrm{V}$ and for the usual values of C and L (see the simulation on [Simulation 13]). The fast oscillations denoted by the solid lines (given by Equation 22) have a frequency of $\bar{\omega} = \frac{\omega_3+\omega_4}{2} \approx 1.24 \times 10^4\,\mathrm{rad/s}$ whose oscillating period is $5.05 \times 10^{-4}\,\mathrm{s}$. On the other hand, the dashed curves represent the modulation envelopes that outline the voltage extremes. The half of the difference in the frequencies is $\delta = \frac{\omega_4-\omega_3}{2} \approx 0.100 \times 10^4\,\mathrm{rad/s}$. For υ_0, υ_2, and υ_4 the envelope period is approximately $6.26 \times 10^{-3}\,\mathrm{s}$, conversely, for υ_1 and υ_3 the period is

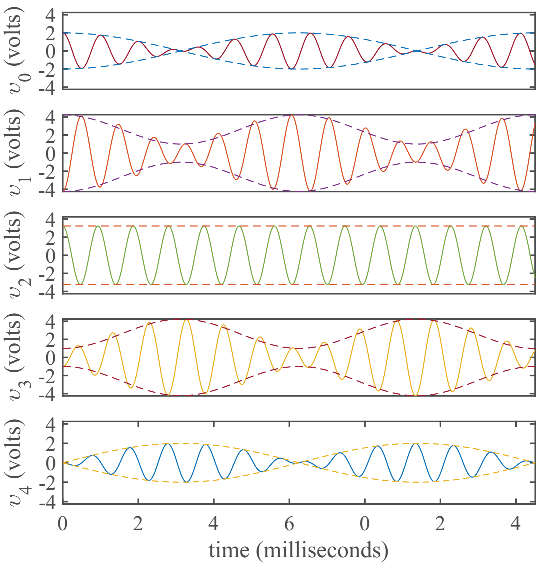

Figure 13: Beatings in a circuit with $N = 4$ loops.

half of the latter. So, even though $N = 4$ is still a small system, one has that $\bar{\omega}$ is approximately 12.4 times larger than the slow frequency δ.

The voltages υ_0, υ_2, and υ_4 behave like the ones in the case $N = 2$. Their modulation envelops are $\upsilon_0 = \pm 2\cos\delta t$, $\upsilon_2 = \pm\text{constant}$, and $\upsilon_4 = \pm 2\sin\delta t$. On the other hand, the oscillations on υ_1 and υ_3 are interpreted as being enveloped by faster oscillations given by $\upsilon_1 \approx \pm\frac{\bar{V}}{2}[3+\sqrt{5}+(1+\sqrt{5})\cos 2\delta t]$ and $\upsilon_3 \approx \pm\frac{\bar{V}}{2}[3+\sqrt{5}-(1+\sqrt{5})\cos 2\delta t]$. Therefore, the upper and lower envelopes that modulate the fast oscillations do not touch each other. Therefore, while the external (υ_0 and υ_4) present nodes in their envelope modulation , the voltages in the middle (υ_1, υ_2, and υ_3) behave like intermediate beats without nodes in their beating pattern.

6. Conclusions

This paper begins with an investigation of two connected LC loops. In general, any oscillation signal is a superposition of the two NFs of the system, on the other hand, special initial conditions can make the loops oscillate with only one of the NFs. This is accomplished by following a simple protocol using external batteries that initially charge the capacitors. Moreover, these situations were simulated using the online platform *Multisim* and both analytical and simulated data were presented. Further, the case of a weak coupling between the loops, oscillations having very close frequencies, and same amplitudes was analyzed. One found that the time evolution of the interference leads to a pattern called beating in acoustic. It is worth pointing out that it is also possible to experimentally explore such circuits, e.g., see Ref. [21], in this case, the dissipative sources might need to be considered.

Then, a periodic LC circuit with N-loops (unit cells) was explored. The set of NFs $\{\omega_n\}$, with $n = 1, 2, \ldots, N$, was found after decoupling the equations for the loop charges. The general behavior was discussed in detail for systems of sizes $N = 3, 4$ and 5. The discussion was focused on how each capacitor needs to be initially charged to synchronize all unit cells oscillating with only one of the NFs. One of the main contributions of this paper is the generalization of the current flows presented in Sec. 4. Also important are the simulations that can be used to explore the coupled oscillations without going through all the mathematical techniques

As discussed in Sec. 5, the larger the circuit, the smaller the difference between two consecutive NFs, for example, the difference between the two fastest frequencies is $(\omega_N - \omega_{N-1}) \propto N^{-2}$, for $N \gg 1$. This fact was used to obtain another important contribution of this work: the emergence of beatings in a homogeneous circuit with $N = 4$. Therefore, an interesting subject for future exploration should be a generalization of how one can prepare an arbitrary sufficiently large system such as one can explore beating along periodic circuits.

Moreover, one can expect that a beat pattern with more than just two frequencies can be created by tuning three or more voltages of equal amplitudes and very close frequencies that would superimpose. Furthermore, one speculates that an interference pattern with a number N_B of close frequencies will show $(N_B - 2)$ secondary peaks/envelopes between the two primary envelopes. The work presented here also opens up room for further exploration of how an external oscillating source could be used to excite each mode ω_n independently [22]. Or even how one could use such a driven oscillation function to tune beats and resonances in large systems.

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